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Nonlinear Variational Inequalities and Implicit Variational Inequality of Ky Fan Type in H-Spaces

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Abstract—In this paper, the character of solution sets for nonlinear variational inequalities and the existence of a solution of implicit variational inequality of Ky Fan type in H-spaces are given.

Keywords—L.c.-space, Locally convex H-space, Interval space, Variational inequality, Monotone function (mapping).

1. INTRODUCTION AND PRELIMINARIES

In 1977, Tarafdar [1] gave the following theorem.

THEOREM A. *Let E be a Hausdorff topological vector space, E^* the conjugate space of E and K a nonempty convex subset of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E^* and E. If $T : K \longrightarrow E^*$ is monotone and hemicontinuous, then a point $u_0 \in K$ is a solution of the variational inequality*

$$\langle T(u_0), v - u_0 \rangle \geq 0, \qquad \forall\, v \in K$$

iff

$$\langle T(v), v - u_0 \rangle \geq 0, \qquad \forall\, v \in K.$$

Using the theorem, Tarafdar [1] proved the existence of solutions of nonlinear variational inequality.

In 1991, Chang [2] generalized the above theorem as follows.

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THEOREM B. *Let X be a nonempty closed convex subset of a Hausdorff topological vector space E, $f : X \to (-\infty, +\infty]$ a lower semicontinuous function with $f \not\equiv +\infty$ and $\phi : X \times X \longrightarrow R$ a monotone hemicontinuous (i.e., for each $y \in X, \phi(\cdot, y)$ is upper semicontinuous on each line-segment in X) function with $\phi(x,x) \geq 0$ for all $x \in X$. If the following conditions are fulfilled:*

(i) *for each $x \in X$, $\phi(x, \cdot)$ is lower semicontinuous;*
(ii) *for each $x \in X$, $f(y) + \phi(x,y)$ is a convex function in y;*

then there exists a point $\bar{x} \in X$ such that

$$f(y) + \phi(\bar{x}, y) \geq f(\bar{x}), \qquad \forall\, y \in X,$$

iff

$$f(y) - \phi(y, \bar{x}) \geq f(\bar{x}), \qquad \forall\, y \in X.$$

As an application of Theorem B, Chang [2] determined the character of solution sets for nonlinear variational inequalities in topological vector spaces.

In the present paper, our purpose is to establish a topological version of Theorems A and B and determine the character of solution sets for nonlinear variational inequalities and the existence of solution of implicit variational inequality of Ky Fan type in H-spaces.

We first give some definitions and notations.

Let X be a topological space and let $\mathcal{F}(X)$ be the family of all nonempty finite subsets of X. Let $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space. Given an H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called H-convex if $\Gamma_A \subset D$ for all $A \in \mathcal{F}(D)$; see [2–6].

An H-space $(X, \{\Gamma_A\})$ is called

(1) a locally convex H-space if X is a uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure $\mathcal{U}$ such that for each $i \in I$, $V_i(x) = \{y \in X : (y,x) \in V_i\}$ is H-convex for each $x \in X$ (see [7]);
(2) an l.c.-space (see [4]) if X is a uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X : E \bigcap V_i[x] \neq \emptyset\}$ is H-convex whenever E is H-convex, where $V_i[x] = \{y \in X : (x,y) \in V_i\}$.

REMARK. The concept of an l.c.-space is different from a locally convex H-space. But an l.c.-space $(X, \{\Gamma_A\})$ with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$ must be a locally convex H-space. A nonempty convex subset X of a locally convex topological vector space is an l.c.-space with $\Gamma_A = \operatorname{co} A$ for all $A \in \mathcal{F}(X)$, and hence, $(X, \{\operatorname{co} A\})$ must be a locally convex H-space.

Let X be an H-space and $f : X \longrightarrow R \bigcup \{+\infty\}$ a function. f is called H-quasiconvex if for each $r \in R$, the set $\{x \in X : f(x) < r\}$ is H-convex; f is called strongly H-quasiconvex if f is H-quasiconvex and for each $r \in R$, if $f(y_1) < r$ and $f(y_2) \leq r$ implies that $f(y) < r$ for all $y \in \Gamma_{\{y_1,y_2\}} \backslash \{y_1, y_2\}$); f is called strictly H-quasiconvex if f is H-quasiconvex and for any $y_1, y_2 \in X$ with $y_1 \neq y_2$, $f(y) < \max\{f(y_1), f(y_2)\}$ for all $y \in \Gamma_{\{y_1,y_2\}} \backslash \{y_1, y_2\}$).

If a topological space X is endowed with a mapping

$$[\cdot,\cdot] : X \times X \longrightarrow \{\text{connected subsets of } X\},$$

such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$, then $(X, [\cdot,\cdot])$ is called an interval space, $[x_1, x_2]$ is called an interval. Obviously, an H-space $(X, \{\Gamma_A\})$ with $\Gamma_{\{x\}} = \{x\} (\forall\, x \in X)$ is an interval space and $[x_1, x_2] = \Gamma_{\{x_1,x_2\}}$ $(x_1, x_2 \in X)$.

Given an interval space $(X, [\cdot,\cdot])$, use w-convex consistently in the whole paper. $f : X \longrightarrow R \bigcup \{+\infty\}$ is called w-convex iff for each $r \in R$, if $f(y_1) < r$ and $f(y_2) \leq r$ implies that $f(y) < r$ for all $y \in [y_1, y_2] \backslash \{y_1, y_2\}$.

Let X be a topological space. We denote by 2^X the family of all subsets of X. If $A \subset X$ we shall denote by $\operatorname{cl}(A)$ the closure of A. If X is a topological vector space, we shall denote the convex hull of A by $\operatorname{co} A$.

A topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus, any convex or star-shaped set is acyclic.

Let X, Y be two topological spaces, $T : X \longrightarrow 2^Y$ a multivalued mapping and $\phi : X \times X \longrightarrow R$ a function.

(1) T is called upper semicontinuous if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$, there exists an open neighborhood U of x such that $T(z) \subset V$ for each $z \in U$.

(2) ϕ is called monotone iff

$$\phi(x,y) + \phi(y,x) \leq 0,$$

for all $(x,y) \in X \times X$.

(3) The multivalued mapping $\mathrm{cl}\, T : X \longrightarrow 2^Y$ is defined by

$$\mathrm{cl}\, T(x) = \mathrm{cl}\,[T(x)], \qquad \forall\, x \in X.$$

Let X be a nonempty convex subset of a Hausdorff topological vector space E, E^* the conjugate space of E ($\langle \cdot, \cdot \rangle$ denotes the pair in duality), and $A : X \longrightarrow E^*$ a mapping.

(1) A is called monotone if $\mathrm{Re}\langle A(x_1) - A(x_2),\, x_1 - x_2 \rangle \geq 0$ for all $x_1, x_2 \in X$.

(2) A is called hemicontinuous if A is continuous from the line-segments in X to the weak topology of E^*.

REMARK. Let $\phi(x,y) = \langle A(x), y - x \rangle$, $x, y \in X$. If A is monotone and hemicontinuous, then ϕ is also monotone and hemicontinuous.

2. A TOPOLOGICAL TOOL

THEOREM 1. *Let $(X, [\cdot,\cdot])$ be a Hausdorff interval space, $f : X \to (-\infty, +\infty]$ a lower semicontinuous function with $f \not\equiv +\infty$, and $\phi : X \times X \longrightarrow R$ a monotone function with $\phi(x,x) \geq 0$ for all $x \in X$. If the following conditions are fulfilled:*

(i) *for each $y \in X$, $\phi(\cdot, y)$ is upper semicontinuous on each interval;*

(ii) *for each $x \in X$, $f(y) + \phi(x,y)$ is w-convex in y;*

then there exists a point $\bar{x} \in X$ such that

$$f(y) + \phi(\bar{x}, y) \geq f(\bar{x}), \qquad \forall\, y \in X,$$

iff

$$f(y) - \phi(y, \bar{x}) \geq f(\bar{x}), \qquad \forall\, y \in X.$$

PROOF. If there exists a point $\bar{x} \in X$ such that

$$f(y) + \phi(\bar{x}, y) \geq f(\bar{x}), \qquad \forall\, y \in X,$$

then

$$f(y) - \phi(y, \bar{x}) \geq f(y) + \phi(\bar{x}, y) \geq f(\bar{x}),$$

for all $y \in X$ since ϕ is monotone. Contrarily, if there exists a point $\bar{x} \in X$ such that

$$f(y) - \phi(y, \bar{x}) \geq f(\bar{x}), \qquad \forall\, y \in X, \tag{1}$$

but, there is a point $\bar{y} \in X$ such that

$$f(\bar{y}) + \phi(\bar{x}, \bar{y}) < f(\bar{x}),$$

then

$$\bar{x} \in U := \{z \in [\bar{x}, \bar{y}] : f(z) - \phi(z, \bar{y}) > f(\bar{y})\}.$$

Obviously, $\bar{y} \notin U$ and U is open in $[\bar{x}, \bar{y}]$ since both f and $-\phi(\cdot, \bar{y})$ are lower semicontinuous on $[\bar{x}, \bar{y}]$.

If $([\bar{x}, \bar{y}] \backslash \{\bar{x}, \bar{y}\}) \bigcap U = \emptyset$, then $U = \{\bar{x}\}$ is closed in $[\bar{x}, \bar{y}]$. It contradicts the connectivity of $[\bar{x}, \bar{y}]$. Hence, there exists a point $\bar{z} \in ([\bar{x}, \bar{y}] \backslash \{\bar{x}, \bar{y}\}) \bigcap U$, i.e., $\bar{z} \in ([\bar{x}, \bar{y}] \backslash \{\bar{x}, \bar{y}\})$ and

$$f(\bar{y}) + \phi(\bar{z}, \bar{y}) < f(\bar{z}). \tag{2}$$

Taking $y = \bar{z}$ in (1) we have

$$f(\bar{x}) + \phi(\bar{z}, \bar{x}) \leq f(\bar{z}). \tag{3}$$

By (ii) together with (2) and (3), we have

$$f(x) + \phi(\bar{z}, x) < f(\bar{z}),$$

for all $x \in [\bar{x}, \bar{y}] \backslash \{\bar{x}, \bar{y}\}$, and hence, $\phi(\bar{z}, \bar{z}) < 0$. It contradicts that $\phi(x, x) \geq 0$ for all $x \in X$. Hence,

$$f(y) + \phi(\bar{x}, y) \geq f(\bar{x}), \qquad \forall\, y \in X.$$

This completes the proof.

COROLLARY 2. *Let X be a nonempty convex subset of a Hausdorff topological vector space E, $f : X \to (-\infty, +\infty]$ a lower semicontinuous function with $f \not\equiv +\infty$ and $\phi : X \times X \longrightarrow R$ a monotone function with $\phi(x, x) \geq 0$ for all $x \in X$. If the following conditions are fulfilled:*

(i) *for each $y \in X$, $\phi(\cdot, y)$ is upper semicontinuous on each line-segment;*
(ii) *for each $x \in X$, $f(y) + \phi(x, y)$ is a convex function in y;*

then there exists a point $\bar{x} \in X$ such that

$$f(y) + \phi(\bar{x}, y) \geq f(\bar{x}), \qquad \forall\, y \in X,$$

iff

$$f(y) - \phi(y, \bar{x}) \geq f(\bar{x}), \qquad \forall\, y \in X.$$

PROOF. For any $x_1, x_2 \in X$, let $[x_1, x_2] = \text{co}\,\{x_1, x_2\}$. Then X is an interval space. Moreover, by (ii) we can prove that $f(y) + \phi(x, y)$ is w-convex in y for each $x \in X$. Consequently, the conclusion follows from Theorem 1.

REMARK. Corollary 2 shows that X need not be closed and $\phi(x, \cdot)$ need not be lower semicontinuous in [2, Theorem 3.9.2] (i.e., Theorem B).

3. SOLUTION SET FOR NONLINEAR VARIATIONAL INEQUALITIES

THEOREM 3. *Let $(X, \{\Gamma_A\})$ be a Hausdorff H-space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, $f : X \longrightarrow R \bigcup \{+\infty\}$ a lower semicontinuous function with $f \not\equiv +\infty$ and $\phi : X \times X \longrightarrow R$ a monotone function with $\phi(x, x) \geq 0$ for all $x \in X$. If the following conditions are fulfilled:*

(i) *there exist a point $x_0 \in X$ and a compact subset K of X such that*

$$f(x) > f(x_0) + \phi(x, x_0), \qquad \forall\, x \in X \backslash K;$$

(ii) *for each $x \in X$, $f(y) + \phi(x, y)$ is strongly H-quasiconvex in y;*
(iii) *for each $y \in X$, $\phi(\cdot, y)$ is upper semicontinuous on $\Gamma_{\{x_1, x_2\}}$ for each $(x_1, x_2) \in X \times X$;*
(iv) *for each $x \in X$, $\phi(x, \cdot)$ is lower semicontinuous;*

then the solution set of the variational inequality

$$\phi(\bar{x}, y) \geq f(\bar{x}) - f(y), \qquad \forall\, y \in X, \tag{$*$}$$

is a nonempty compact H-convex subset of K.

If (ii) is replaced by the following:

(ii)′ *for each $x \in X$, $f(y) + \phi(x, y)$ is strictly H-quasiconvex in y,*

then the variational inequality $(*)$ has a unique solution $\bar{x}$ and $\bar{x} \in K$.

PROOF. For any $x_1, x_2 \in X$, let $[x_1, x_2] = \Gamma_{\{x_1, x_2\}}$. Then X is an interval space. For each $y \in X$, let

$$M(y) = \{x \in X : f(y) + \phi(x, y) \geq f(x)\}, \qquad H(y) = \{x \in X : f(y) - \phi(y, x) \geq f(x)\}.$$

Then $H(y)$ is closed and H-convex by (ii), (iv), and the lower semicontinuity of f. Let

$$S = \bigcap_{y \in X} M(y).$$

Then S is the set of all solutions of the variational inequality $(*)$. By Theorem 1 we know that $S = \bigcap_{y \in X} H(y)$.

For each finite subset $\{y_1, y_2, \dots, y_n\} \subset X$, if $x \notin \bigcup_{i=1}^{n} M(y_i)$, then $f(y_i) + \phi(x, y_i) < f(x)$ for all $i \in \{1, 2, \dots, n\}$. Since $f(y) + \phi(x, y)$ is H-quasiconvex in y,

$$\Gamma_{\{y_1, y_2, \dots, y_n\}} \subset \{y \in X : f(y) + \phi(x, y) < f(x)\}$$

so that $x \notin \Gamma_{\{y_1, y_2, \dots, y_n\}}$ because $\phi(x, x) \geq 0$. This shows that $M : X \longrightarrow 2^X$ is an H-KKM mapping so that $\operatorname{cl} M : X \longrightarrow 2^X$ is also an H-KKM mapping. Moreover, by (i) we know that $M(x_0) \subset K$, and hence, $\operatorname{cl}[M(x_0)] \subset K$ is compact. By virtue of the H-KKM theorem (see also [2–4]), we know that

$$\bigcap_{y \in X} \operatorname{cl}\,[M(y)] \neq \emptyset.$$

Noting that $\operatorname{cl}[M(y)] \subset H(y)$ for all $y \in X$, we know that

$$S = \bigcap_{y \in X} H(y) \neq \emptyset.$$

Consequently, S is a nonempty compact H-convex subset of K since

$$\bigcap_{y \in X} H(y) = S \subset \operatorname{cl} S \subset \operatorname{cl}\,[M(x_0)] \subset K$$

and $H(y)$ is closed and H-convex for all $y \in X$.

If (ii) is replaced by (ii)′, then for any $x_1, x_2 \in S$, we have

$$f(y) + \phi(x_i, y) \geq f(x_i), \qquad \forall\, y \in X, \tag{4}$$

$i = 1, 2$. Since ϕ is monotone and $\phi(x, x) \geq 0$, $\phi(x, x) = 0$ for all $x \in X$. If $x_1 \neq x_2$, taking $y \in \Gamma_{\{x_1, x_2\}} \backslash \{x_1, x_2\}$ in (4) we have

$$f(x_1) \leq f(y) + \phi(x_1, y) < f(x_2) + \phi(x_1, x_2)$$

and

$$f(x_2) \leq f(y) + \phi(x_2, y) < f(x_1) + \phi(x_2, x_1)$$

by (ii)′.

Consequently,

$$\phi(x_1, x_2) > f(x_1) - f(x_2)$$

and

$$\phi(x_2, x_1) > f(x_2) - f(x_1),$$

and hence,

$$\phi(x_1, x_2) + \phi(x_2, x_1) > 0.$$

It contradicts the monotoneity of ϕ. Therefore, S contains a unique element; i.e., the variational inequality $(*)$ has a unique solution $\bar{x}$ and $\bar{x} \in K$.

COROLLARY 4. *Let X be a nonempty closed convex subset of a Hausdorff topological vector space E, $f : X \longrightarrow R$ a lower semicontinuous function and $\phi : X \times X \longrightarrow R$ a monotone function with $\phi(x,x) \geq 0$ for all $x \in X$. If the following conditions are fulfilled:*

(i) *there exist a compact subset K of E and a point $x_0 \in X$ such that*

$$f(x) > \phi(x, x_0) + f(x_0), \qquad \forall x \in X \backslash K;$$

(ii) *for each $x \in X$, $f(y) + \phi(x,y)$ is a convex function in y;*
(iii) *for each $y \in X$, $\phi(\cdot, y)$ is upper semicontinuous on each line-segment in X;*
(iv) *for each $x \in X$, $\phi(x, \cdot)$ is lower semicontinuous;*

then the solution set of the variational inequality

$$\phi(x,y) \geq f(x) - f(y), \qquad \forall y \in X, \tag{$*$}$$

is a nonempty compact convex subset of $X \bigcap K$.

If (ii) is replaced by the following:

(ii)$'$ *for each $x \in X$, $f(y) + \phi(x,y)$ is strictly convex in y,*

then the variational inequality $(*)$ has a unique solution $\bar{x}$ and $\bar{x} \in X \bigcap K$.

PROOF. For each finite subset A of X, let $\Gamma_A = \operatorname{co} A$. Then $(X, \{\operatorname{co} A\})$ is a Hausdorff H-space. Since $\phi(x,x) \geq 0$ for all $x \in X$, $x_0 \in X \bigcap K$ by (i). Since again X is closed in E, $X \bigcap K$ is a nonempty compact subset of X. Note that $X \backslash K = X \backslash (X \bigcap K)$. By (i) we know that

$$f(x) > \phi(x, x_0) + f(x_0), \qquad \forall x \in X \backslash \left(X \bigcap K \right).$$

Consequently, the corollary follows from Theorem 3.

REMARK. Corollary 4 shows that $\phi(\cdot, y)$ need not be upper semicontinuous in [2, Theorem 3.9.3].

COROLLARY 5. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and E^* the conjugate space of E ($\langle \cdot, \cdot \rangle$ denotes the pair in duality). Let $A : X \longrightarrow E^*$ be monotone and hemicontinuous. Then the solution set of the variational inequality*

$$\operatorname{Re}\langle Ax, y - x \rangle \geq 0, \qquad \forall y \in X, \tag{$*$}$$

is a nonempty compact convex subset of X.

PROOF. For each $(x,y) \in X \times X$, let $\phi(x,y) = \operatorname{Re}\langle Ax, y - x \rangle$. Then $\phi : X \times X \to R$ satisfies all conditions of Corollary 4. Consequently, the corollary follows from Corollary 4.

REMARK. Corollary 5 is finer than the corollary in [1].

4. KY FAN TYPE IMPLICIT VARIATIONAL INEQUALITY IN H-SPACES

First, we give a lemma, which appeared in [7].

LEMMA 6. *Let $(X, \{\Gamma_A\})$ be a compact Hausdorff locally convex H-space and $T : X \longrightarrow 2^X$ an upper semicontinuous multivalued mapping with closed acyclic values. Then there exists a point $x_0 \in X$ such that $x_0 \in T(x_0)$.*

THEOREM 7. *Let $(X, \{\Gamma_A\})$ be a compact Hausdorff l.c.-space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, $g : X \times X \longrightarrow R \bigcup \{+\infty\}$ a lower semicontinuous function and $\psi : X \times X \times X \longrightarrow R$ a function with $\psi(z,x,x) \geq 0$ for all $(z,x) \in X \times X$. If the following conditions are fulfilled:*

(i) *for each $z \in X$, there exists a point $x \in X$ such that $g(z,x) < +\infty$;*
(ii) *for each $z \in X$, $\phi(x,y) = \psi(z,x,y)$ is monotone;*
(iii) *for each $x \in X$, $g(\cdot, x)$ is upper semicontinuous;*
(iv) *for each $(z,x) \in X \times X$, $g(z,y) + \psi(z,x,y)$ is strictly H-quasiconvex in y;*
(v) *$\psi(z, \cdot, y)$ is upper semicontinuous on each $\Gamma_{\{x_1,x_1\}} (x_1, x_1 \in X)$ and $\psi(\cdot, x, \cdot)$ is lower semicontinuous;*

then there exists a point $\bar{x} \in X$ such that

$$g(\bar{x}, y) + \psi(\bar{x}, \bar{x}, y) \geq g(\bar{x}, \bar{x}), \qquad \forall\, y \in X.$$

PROOF. For any x_1, $x_2 \in X$, let $[x_1, x_2] = \Gamma_{\{x_1, x_2\}}$. Then $(X, [\cdot,\cdot])$ is a compact Hausdorff interval space. For each fixed $z \in X$, let $f(x) = g(z, x)$ for all $x \in X$. Then f, ϕ satisfies all conditions of Theorem 1. By Theorem 1, $\bar{x} \in X$ is a solution of the variational inequality

$$f(y) + \phi(x, y) \geq f(x), \qquad \forall\, y \in X, \tag{5}$$

iff $\bar{x} \in X$ is a solution of the variational inequality

$$f(y) - \phi(y, x) \geq f(x), \qquad \forall\, y \in X. \tag{6}$$

Note that f, ϕ satisfy all conditions of the second part of Theorem 3. By virtue of Theorem 3 we know that the variational inequality (5) has unique solution $s(z)$, and hence, the variational inequality (6) has unique solution $s(z)$.

Let $\{(z_\alpha, s(z_\alpha) : \alpha \in I\}$ be a net converging to (z, x). Then for each $y \in X$, we have

$$g(z_\alpha, y) - \psi(z_\alpha, y, s(z_\alpha)) \geq g(z_\alpha, s(z_\alpha))$$

for all $\alpha \in I$. By (iii), (v), and the lower semicontinuity of g, we know that

$$\begin{aligned} g(z, x) &\leq \underline{\lim}_\alpha g(z_\alpha, s(z_\alpha)) \\ &\leq \underline{\lim}_\alpha [g(z_\alpha, y) - \psi(z_\alpha, y, s(z_\alpha))] \\ &\leq \overline{\lim}_\alpha [g(z_\alpha, y) - \psi(z_\alpha, y, s(z_\alpha))] \\ &\leq g(z, y) - \psi(z, y, x), \end{aligned}$$

for all $y \in X$. This shows that $x = s(z)$, and hence, the graph of s is closed and $s : X \longrightarrow X$ is a single-valued upper semicontinuous mapping.

Note that $(X, \{\Gamma_A\})$ is also a compact Hausdorff locally convex H-space. By virtue of [7, Theorem 1], i.e., Lemma 6, there exists a point $\bar{x} \in X$ such that $\bar{x} = s(\bar{x})$, i.e.,

$$g(\bar{x}, y) + \psi(\bar{x}, \bar{x}, y) \geq g(\bar{x}, \bar{x}), \qquad \forall\, y \in X.$$

This completes the proof.

COROLLARY 8. *Let $(X, \{\Gamma_A\})$ be a compact Hausdorff l.c.-space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$ and $\psi : X \times X \times X \longrightarrow R$ a function with $\psi(z, x, x) \geq 0$ for all $(z, x) \in X \times X$. If the following conditions are fulfilled:*

- (i) *for each $z \in X$, $\phi(x, y) = \psi(z, x, y)$ is monotone;*
- (ii) *for each $(z, x) \in X \times X$, $\psi(z, x, y)$ is strictly H-quasiconvex in y;*
- (iii) *$\psi(z, \cdot, y)$ is upper semicontinuous on each $\Gamma_{\{x_1, x_1\}} (x_1, x_1 \in X)$ and $\psi(\cdot, x, \cdot)$ is lower semicontinuous;*

then there exists a point $\bar{x} \in X$ such that

$$\psi(\bar{x}, \bar{x}, y) \geq 0, \qquad \forall\, y \in X.$$

PROOF. Take $g \equiv 0$ in Theorem 7. Then the conclusion follows from Theorem 7.

REMARK. Corollary 8 extends Ky Fan's theorem for implicit variational inequalities (see also [2, Theorem 6.7.1]) to H-spaces. Theorem 7 improves [2, Theorem 6.6.1].

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